Scaling theory and spreading dynamics in systems with one absorbing state derived from an equilibrium statistical model

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We show that for systems with one absorbing state, the widely assumed scaling properties of the survival probability and of the probability density of the size of activity avalanches cannot be true in the asymptotic limit. Trying to answer the question, what is the true asymptotic limit of these quantities, we study Domany-Kinzel probabilistic cellular automata using an equilibrium statistical mechanic model (ESM). We are able to express important quantities of the avalanche dynamics by correlation functions of the ESM. The application of scaling theory to the ESM allows for the derivation of the scaling properties of quantities of the avalanche dynamics in the form of infinite series. From these results we can obtain possible solutions for the apparent scaling problem, but cannot decide definitely which one is true. The most appealing solution, for which some evidence is given, states that there is a narrow range around the critical point in which, for example, the survival probability has the same power-law behavior as on the critical point. Outside this narrow range, the usually assumed scaling should be approximately valid.

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I. INTRODUCTION

Stochastic nonequilibrium systems with absorbing states, i.e., states in which the system can be trapped without any fluctuations, and with phase transitions from the absorbing states to active states are important models for describing real world processes such as the spread of epidemics, percolation processes, autocatalytical reactions, and many more, for an overview see Ref. [1]. A canonical example of such systems with one absorbing state is the directed percolation (DP) universality class. This DP class has been extensively investigated in the past [1–3,8,19] and it seems that we have now a full understanding of these systems. However, in this paper, we show that there are problems left concerning the scaling in the vicinity of the critical point of certain probability distributions describing the dynamic of the spreading of activity avalanches generated from a localized "seed" of activity in the absorbing state. An example is the survival probability $P(t)$ of avalanches and the probability density $p(s)$ of the size of avalanches. First, we show exactly that the usually assumed scaling of $P(t) \sim t^{-\delta} g(\Delta t^{1/\nu_\parallel})$ and $p(s) \sim s^{-\tau} f(\Delta s^{\sigma})$, where Δ is the distance from the critical point and δ, $\nu_\parallel$, τ, σ are universal critical exponents, cannot be true in the asymptotic limit $t \to \infty$, $s \to \infty$, $\Delta \to 0$. We present simulation results for a four-dimensional Domany-Kinzel probabilistic cellular automaton (PCA) [4,5] that exhibit relative clearly that there is no conventional scaling for $P(t)$. Indeed, up to now, the theoretical foundation for the assumed scaling of $P(t)$ and $p(s)$ is not very strong. The origin of the scaling of other quantities of the spreading dynamic of avalanches, such as the mean number of active sites in avalanches or the mean spatial extension of avalanches [6,7], is well understood from the renormalization of the field theoretical formulation of the system model continuous in space and time [8,9]. This is the case, because these quantities can be expressed by correlation functions of the continuous fields for which the scaling properties are well established from the renormalization group approach. The continuous field approach is not capable to describe the avalanche dynamic, this is mainly the case because the absorbing state is a set of measure zero [10]. Therefore, to investigate theoretically what the correct asymptotic behavior of $P(t)$ and $p(s)$ near the critical point is, we must use the original discrete space structure of the PCA on a lattice and describe it by a Markov process continuous or discontinuous in time [1,11].

In the second part of the paper we use a description as a Markov process discontinuous in time together with the long-known fact that the d-dimensional Domany-Kinzel PCA can be mapped to an equivalent equilibrium statistical mechanic model (ESM) described by a Hamiltonian on a discrete space-time lattice [4,5,12,14]. We are able to express the survival probability $P(t)$ and the mean life-time D_A of avalanches as series over certain correlation functions of the corresponding ESM. Applying the principles of the renormalization group to the ESM Hamiltonian, we get the scaling properties of the ESM correlation functions. Using these properties the behavior of $P(t)$ for large t near the critical point is obtained in the form of an infinite series. Also, for D_A we get an infinite series of certain power-law terms. This second part of the paper is somewhat cumbersome, but we think that it is necessary to review the mapping of a PCA to an ESM in some details to be able to derive the meaning of the ESM correlation functions and the scaling properties of these functions. These results are central to our approach and are not presented in the literature up to now in sufficient detail. From the derived series expansion of $P(t)$ and D_A, we are able to point out the possibility that there is a narrow range around the critical point, where the scaling of $P(t)$ is the same as on the critical point, namely $P(t) \sim t^{-\delta}$, and the conventional scaling is only approximately valid outside this narrow range. We cannot prove this very unusual view exactly, but we present some evidence from theoretical considerations and from simulation results that it may be true. If we

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are wrong, there is only the possibility that $P(t)$ obeys no universal scaling properties asymptotically, and this is not very satisfactory.

II. ANOMALOUS SCALING OF THE SURVIVAL PROBABILITY

Consider a PCA from the DP universality class defined on an infinite discrete space lattice. Each lattice site can be in an active or inactive state. The time evolution is considered to be continuous or discontinuous. Setting a site active in the absorbing state results in the development of an avalanche of activation. If the system parameters are such that no active phase exists, the avalanche is always dumped out in a finite time and the absorbing state is reached again. In the case that an active phase exists, there is some probability that the active state is reached for time $t\to\infty$. For a definition and discussion of the critical exponents, the scaling properties of quantities used in the description of the spreading dynamics and for the derivation of scaling relations between the critical exponents, see Refs. [1,6,7]. The survival probability $P(t)$ of activity avalanches is the probability that at time t the absorbing state is not reached. The usually assumed asymptotic scaling behavior of $P(t)$ in the vicinity of the critical point is

$$P(t)\sim\frac{1}{t^{\delta}}g(t^{1/\nu_\parallel}\Delta), \tag{1}$$

where Δ is the distance from the critical point, we choose $\Delta<0$ in the inactive phase and $\Delta>0$ in the active phase, and δ, $\nu_\parallel$ are universal critical exponents. The probability density of $P(t)$ is $p(t)=-\partial P(t)/\partial t$ for large t. From Eq. (1) we obtain

$$p(t)\sim\frac{1}{t^{1+\delta}}h(t^{1/\nu_\parallel}\Delta). \tag{2}$$

The exact meaning of Eq. (2) is

$$\lim_{t\to\infty,\Delta\to 0}\frac{p(t,\Delta)}{p_a(t,\Delta)}=1, \tag{3}$$

where $p(t,\Delta)$ is the real probability density and $p_a(t,\Delta)$ is the right-hand side of Eq. (2). Note that Eq. (2) does not determine $p_a(t,\Delta)$ uniquely, it is only a necessary condition. For a discontinuous PCA, functions such as $p(t,\Delta)$ are only defined on discrete time steps. In this case, we extend the definition to continuous times using staircase functions, then all summations over discrete times can be expressed by integrations over time. The survival probability is $P(t,\Delta)=1-\int_0^t p(u,\Delta)du$, and the normalization condition of $p(t,\Delta)$ reads $\int_0^\infty p(t,\Delta)dt=1-P(\infty,\Delta)$, where $P(\infty,\Delta)=0$ for $\Delta<0$. For the following consideration, it is important that for finite times it is possible to write $p(t,\Delta)=\sum_{n=0}^\infty p_n(t)\Delta^n$. This must be true for a discontinuous PCA, because the update rules ensure that the probability of each possible activation pattern occurring at a given time is a polynomial of the probability parameters of the update rules. A continuous PCA can be considered as the limit of a sequence of discontinuous PCA's with time-step differences going to zero, so that in this case, $p(t,\Delta)$ must also be a power series in Δ. It is safe to write $P(t,\Delta)=1-\sum_{n=0}^\infty[\int_0^t p_n(u)du]\Delta^n$, showing that $P(t,\Delta)$ is also a power series in Δ for finite times. We are now able to prove that the Eq. (3) cannot be valid. The crucial points are that the condition $\int_0^\infty p(t,\Delta)dt=1$ must be fulfilled for $\Delta<0$ independent of Δ and that $p(t,\Delta)$ can be expanded in a power series of Δ.

Suppose that Eq. (3) is valid, we can write

$$p(t,\Delta)=p_a(t,\Delta)+\varepsilon(t,\Delta), \tag{4}$$

and choose a t_0 and a $|\Delta_0|$ such that for $t>t_0$ and $|\Delta|<|\Delta_0|$ it is $|\varepsilon(t,\Delta)|\ll p_a(t,\Delta)$ and

$$\left|\int_{t_0}^\infty\varepsilon(t,\Delta)dt\right|\ll\int_{t_0}^\infty p_a(t,\Delta)dt. \tag{5}$$

Further, we conclude that $p_a(t,\Delta)$ is also a power series in Δ. The normalization condition now reads

$$1-P(\infty,\Delta)-\sum_{n=0}^\infty c_n\Delta^n=\int_{t_0}^\infty p_a(t,\Delta)dt+\int_{t_0}^\infty\varepsilon(t,\Delta)dt, \tag{6}$$

where $c_n=\int_0^{t_0}p_n(t)dt$. For $\Delta\neq 0$ and after substituting $u=|\Delta|t^{1/\nu_\parallel}$, the first integral in Eq. (6) is

$$\int_{t_0}^\infty p_a(t,\Delta)dt=\nu_\parallel|\Delta|^{\nu_\parallel\delta}\left(\int_{|\Delta|t_0^{1/\nu_\parallel}}^{u_0}\frac{h^{\pm}(u)}{u^{1+\nu_\parallel\delta}}du+\int_{u_0}^\infty\frac{h^{\pm}(u)}{u^{1+\nu_\parallel\delta}}du\right), \tag{7}$$

with the constant $u_0>|\Delta_0|t_0^{1/\nu_\parallel}$ and $h^{\pm}(u)$ being the parts of h for $\Delta>0$ and $\Delta<0$. Consider the right-hand side of Eq. (7), the second integral is a constant independent of Δ, further inserting $h^{\pm}(u)=\sum_{n=0}^\infty h_n^{\pm}u^n$ into the first integral and integrating all terms, we obtain

$$\int_{t_0}^\infty p_a(t,\Delta)dt=\frac{h^{\pm}(0)}{\delta t_0^{\delta}}-\nu_\parallel a^{\pm}|\Delta|^{\nu_\parallel\delta}+\sum_{n=1}^\infty a_n^{\pm}|\Delta|^n \tag{8}$$

for $d=1,2,3$, where $\nu_\parallel\delta<1$ [6], $a^{\pm}=\int_0^\infty du[h^{\pm}(0)-h^{\pm}(u)]/u^{1+\nu_\parallel\delta}\neq 0$, and

$$\int_{t_0}^\infty p_a(t,\Delta)dt=\frac{h^{\pm}(0)}{t_0}-h_1^{\pm}|\Delta|\ln(|\Delta|)+\sum_{n=1}^\infty b_n^{\pm}\Delta^n \tag{9}$$

for $d\geqslant 4$, with $\nu_\parallel=\delta=1$ [6]. The $a_n^{\pm}$ and $b_n^{\pm}$ are constants independent of Δ. From this result it follows that for $\Delta<0$, where $P(\infty,\Delta)=0$, and $\Delta\to 0$ there is a singularity (for $d\geqslant 4$ we assume $h_1^{\pm}\neq 0$), so that $\int_{t_0}^\infty p_a(t,\Delta)dt$ cannot be expanded in a power series of Δ around $\Delta=0$, and because of Eq. (5), this singularity cannot be canceled out by a possible similar singularity of $\int_{t_0}^\infty\varepsilon(t,\Delta)dt$. The conclusion is that Eq.

(6) cannot be satisfied, thus, the scaling of $p(t,\Delta)$ is not possible for $\Delta<0$ in the sense of Eqs. (3) and (2). Further, the scaling of the survival probability given by Eq. (1) cannot be true. An exception can be the $d\geqslant 4$ case, where scaling is possible if the scaling function is such that $h_1^{\pm}=0$, but that seems very unlikely. The density of avalanche sizes $p(s,\Delta)$ also cannot scale for $\Delta<0$ in the usual form given above. This follows from the fact that $p(s,\Delta)$ is also expandable as a power series in Δ, and the above derivation is valid with minor changes if $\nu_\parallel$ is replaced by $1/\sigma$ and δ by $\tau-1$. Note that the above arguments do not apply to quantities for which no normalization condition is needed, for example, not to the mean number of active sites $I(t,\Delta)\sim t^{\eta}\phi(t^{1/\nu_\parallel}\Delta)$ [6] in activity avalanches at a given time.

We have shown that for $\Delta<0$ the usual assumed scaling is not possible for $p(t,\Delta)$ and $P(t,\Delta)$. Is this also true for $\Delta>0$? From the above discussion we cannot infer this, because now $P(\infty,\Delta)>0$ and $P(\infty,\Delta)$ may have the correct singularity for $\Delta\rightarrow 0$ to satisfy the Eq. (6). Indeed, the scaling assumption Eq. (1) can be written as $P(t,\Delta)\sim\Delta^{\nu_\parallel\delta}\tilde{g}(t^{1/\nu_\parallel}\Delta)$, so it seems that $P(\infty,\Delta)\sim\Delta^{\nu_\parallel\delta}$ [1,7], and at least for $d=1,2,3$ this is the needed singularity. But this conclusion is wrong as is now shown: $P(t,\Delta)$ can be expanded in power series of Δ, therefore, also $g(u)$, where $u=t^{1/\nu_\parallel}\Delta$. To get $P(\infty,\Delta)\sim\Delta^{\nu_\parallel\delta}$ we need $g(u)\sim u^{\nu_\parallel\delta}$ for $u\rightarrow\infty$. Writing $g(u)=cu^{\nu_\parallel\delta}+\varepsilon(u)$ shows that there must exist a $u_0>0$ such that for $u>u_0$ it is $|\varepsilon(u)|\ll cu^{\nu_\parallel\delta}$, so $\varepsilon(u)$ cannot cancel $cu^{\nu_\parallel\delta}$. Thus, it is necessary that $u^{\nu_\parallel\delta}$ can be expanded in a unique power series in the range $u_0<u<\infty$ to fulfill the last equation, but this is not possible if $\nu_\parallel\delta$ is not an integer. For $d=1,2,3$ $\nu_\parallel\delta$ is not an integer, so the scaling assumption cannot be true. For $d\geqslant 4$, this is also the case, because $P(\infty,\Delta)$ has not the needed singularity $\Delta\ln\Delta$ of Eq.(9). Later it is shown that $P(\infty,\Delta)\sim\Delta^{\nu_\parallel\delta}$ is indeed true, but this can only be realized if the scaling assumption is not true. Then in $P(t,\Delta)=\Sigma_{n=0}^{\infty}P_n(t)\Delta^n$ the $P_n(t)$ can go in the limit $t\rightarrow\infty$ to $\pm\infty$, so that it is not possible to calculate $\lim_{t=\infty}P(t,\Delta)$ by applying $\lim_{t=\infty}$ to each term of the series, and thus, it can be that $P(\infty,\Delta)$ is not a power series in Δ.

Now the question arises what the correct asymptotic behavior of $p(t,\Delta)$ and $p(s,\Delta)$ really is. From the above result we can only conclude that $p(x,\Delta)\sim x^{-\varepsilon}f(x,\Delta)$, where $x=t,s$, $\varepsilon=1+\delta,\tau$ and $f(x,\Delta)$ must be a power series in Δ for finite x and $\lim_{x=\infty}f(x,0)>0$. Before we try to answer this question, we present some simulation results, which show relatively clearly that the scaling of $P(t,\Delta)$ for some cases is not even fulfilled approximately. We have performed extensive simulations of the Domany-Kinzel PCA [4,5] for the dimensions $d=1$ and $d=4$ on a hypercubic lattice, and with the update rule that a site is activated with probability p if at least one nearest neighbor is active. The results show that for the case $d=1$ in the inactive phase and in the range $|\Delta|t^{1/\nu_\parallel}=0.005,\ldots,0.06$, corresponding to maximal avalanche durations up to $t=50\,000$ and $10^4|\Delta|=0.5$, 1.0, 2.0, the scaling of $P(t,\Delta)$ is nearly as good as of $I(t,\Delta)$. For this simulation, the highly accurate values for p_C, $\nu_\parallel$, η, and δ obtained by Jensen [16] from series expansions are used. According to the above result, the accuracy of scaling should

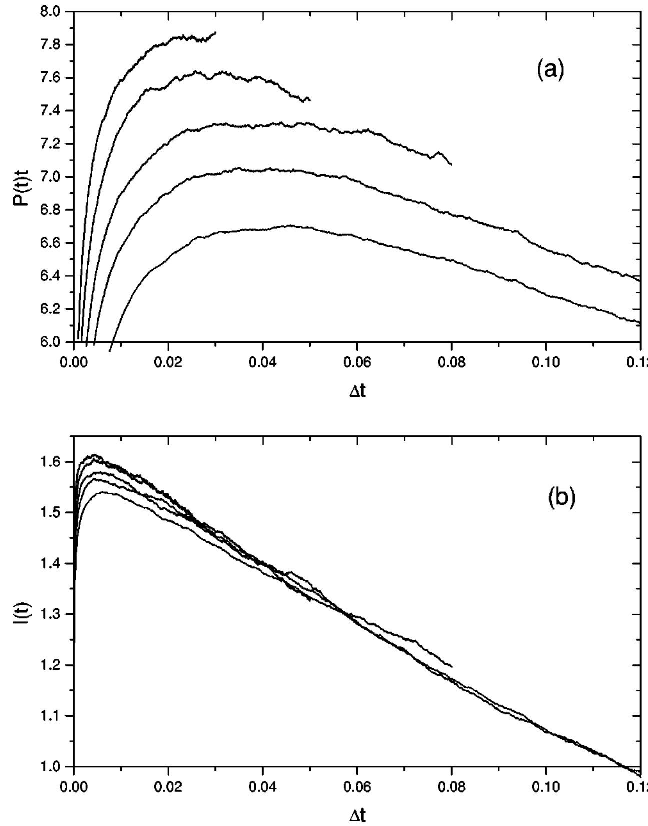

FIG. 1. Scaling of the survival probability (a) and the activity intensity (b) for a four-dimensional Domany-Kinzel PCA near the critical point $p_c=0.146\,158\,0(5)$ for $10^5\Delta=0.3$, 0.5, 0.8, 1.3, and 2.3 (from above to below) in the inactive phase.

break down for larger $|\Delta|t^{1/\nu_\parallel}$ values and the same or lesser Δ values, but these values are not reachable for simulations. But for $d=4$, the situation is completely different, in the inactive phase and the range $|\Delta|t^{1/\nu_\parallel}=0.025,\ldots,0.12$, corresponding to maximal avalanche duration $t=10\,000$ and $10^5|\Delta|=0.3$, 0.5, 0.8, 1.3, 2.3, the scaling of $P(t,\Delta)$ is very bad compared to the $I(t,\Delta)$ scaling, as shown in Fig. 1. The p_C value has been determined similar as described by Voigt and Ziff [19] from a plot of $I(t)/t^{\eta}$ versus $t^{1/\nu_\parallel}$, with $\eta=0$ and $\nu_\parallel=1$, for different Δ values. The obtained result is $p_C=0.146\,158\,0(5)$. The sample size for each Δ value has been about 10^7 avalanches, resulting in a statistical error considerably lower than the scaling violation of $P(t,\Delta)$. It would be tempting to attribute the bad scaling of $P(t,\Delta)$ to a large correction to scaling, but Fig. 1 does not show that the scaling violation goes to zero for $\Delta\rightarrow 0$, therefore, we conclude that the result indicates that $h_1^{\pm}\neq 0$, and thus, there is no exact scaling.

The above results can be extended to systems in the DP class with nonuniversal exponent δ, especially systems with many absorbing states [1,17]. The only requirement is that $p(t,\Delta)$ is expandable in a power series of Δ, but this should be the case. The anomalous scaling of avalanche quantities shown above has some similarities with situations in other contexts. For example, in the theory of kinetic roughening, it is known that the scaling function of the local interface width is not always an analytical function of a quantity that has a meaning analogous to $t^{1/\nu_\|}\Delta$, for details see [18]. But in our case, the situation is much more serious, it is $p(t,\Delta) \sim t^{-\delta}f(t,\Delta)$ and it is not possible that f is only a function of $t^{1/\nu_\|}\Delta$. In the following part of the paper we try to answer the now apparent scaling problem of quantities important for the description of activity avalanches. As explained in the Introduction, discrete lattice models must be used to get inside in the dynamic of activity avalanches.

III. MAPPING OF A PCA TO AN EQUILIBRIUM STATISTICAL MECHANICAL MODEL

We consider PCA models on discrete d-dimensional lattices of N sites, which evolve in discrete time steps $t = 0,1,2,\ldots,$ and where each lattice site i at time t can be only in an inactive state $s_i^t = 0$ or in an active state $s_i^t = 1$. The global state of the PCA at time t is given by $I = (s_1^t, s_2^t, \ldots, s_N^t) = \{s_i^t\}$. It is assumed that the time evolution of the PCA is a Markov process

$$\mathbf{P}(t+1) = \mathbf{T}\mathbf{P}(t) = \mathbf{T}^{t+1}\mathbf{P}(0), \tag{10}$$

where $\mathbf{P}(t) = [P_I(t)]$ is the column vector of the probabilities $P_I(t)$ that at time t the PCA is in the state I, and $\mathbf{T} = (T_{I,J})$ is the matrix of transition probabilities from state J to state I. The update rules of the PCA are assumed to be such that there are local probabilities $p(s_i^{t+1}|\mathbf{n}_i^t)$, so that

$$T_{I,J} = \prod_{i=1}^{N} p(s_i^{t+1}|\mathbf{n}_i^t), \tag{11}$$

with $I = \{s_i^{t+1}\}$ and $J = \{s_j^t\}$, and $\mathbf{n}_i^t = (s_{j_1}^t, s_{j_2}^t, \ldots, s_{j_K}^t)$ describes all possible states of K sites from a neighborhood U of site i. To preserve the total probability $p(0|\mathbf{n}_i^t) + p(1|\mathbf{n}_i^t) = 1$ is required. For later use, some well-known results about Markov processes are listed. If λ_α are the eigenvalues, $\boldsymbol{\phi}_\alpha^R$ (column vector) the right eigenvectors and $\boldsymbol{\phi}_\alpha^L$ (row vector) the left eigenvectors of $\mathbf{T}$, the spectral representation of $\mathbf{T}^t$ is $\mathbf{T}^t = \Sigma_\alpha(\lambda_\alpha)^t\mathbf{T}_\alpha$, with $\mathbf{T}_\alpha = \boldsymbol{\phi}_\alpha^R\boldsymbol{\phi}_\alpha^L$ and $\boldsymbol{\phi}_\alpha^L\boldsymbol{\phi}_\beta^R = \delta_{\alpha\beta}$. Assuming the ergodic condition $(\mathbf{T}^t)_{I,J} > 0$ for some t and all I,J, a unique eigenvalue $\lambda_\alpha = 1$ with $\boldsymbol{\phi}_\alpha^L = (1,1,\ldots,1)$ will exist and the other eigenvalues are $|\lambda_\beta| < 1$. In this case the PCA evolves to the stationary state $\mathbf{P}(\infty) = \boldsymbol{\phi}_\alpha^R$ independent of the system history. If the PCA is not ergodic or the system size N goes to infinity, there can be more than one eigenvalue $\lambda_\alpha = 1$ [14]. For $N \to \infty$ a hypersurface can exist in the space Ω of parameters defining the local transition probabilities p on which the number of eigenvalues $\lambda_\alpha = 1$ change. This hypersurface separates different phases of the system. Consider a point of Ω with more than one eigenvalues $\lambda_\alpha = 1$: more than one stationary state (attractor) will exist and the selected one depending on the initial condition $\mathbf{P}(0)$.

From the Markov process, an equivalent equilibrium statistical mechanic model (ESM) can be obtained. For detailed properties of the ESM and for applications see Refs. [4,5,12,14]. We now review the construction of the ESM. A path of states $I_0, I_1, \ldots, I_{M-1}$ evolving from $t = 0$ to $t = M-1$ can be considered as the configurations of a $d+1$ dimensional lattice. Using Eqs. (10) and (11) the probability of such a history path is given by

$$p_{Path} = T_{I_{M-1},I_{M-2}} \cdots T_{I_2,I_1}T_{I_1,I_0} = \exp[-H(s_t^t)], \tag{12}$$

where

$$H(s_t^t) = -\sum_{i=1}^{N}\sum_{t=0}^{M-1}[\ln\{p(s_i^{t+1}|\mathbf{n}_i^t)\} + h_i^t s_i^t], \tag{13}$$

and the h_i^t are auxiliary fields that go finally to zero. This construction can be viewed as a $d+1$-dimensional system defined on a discrete lattice in thermodynamic equilibrium with an Hamiltonian $H(s_t^t)$ and a transfer matrix $\mathbf{T}$. The partition function $Z(M,N)$ of this finite ESM depends on the boundary conditions adopted for the $d+1$-dimensional lattice. Choosing the $t=0$ boundary (state I_0) as fixed and summing over all other site states $s_i^t = 0,1$ yields $Z = 1$ independent of I_0 [13]. This is simply a consequence of $\Sigma_I T_{I,J} = 1$. The drawback of this boundary condition is that formally for each I_0 there is a different Hamiltonian. To circumvent this we simply sum over all I_0 states yielding $Z = 2^N$. Another possibility choosing the boundary conditions is $I_0 = I_{M-1}$ (periodic boundary conditions) [14]. Using the spectral representation of T^M we get for this case

$$Z(M,N) = \sum_I (T^M)_{I,I} = \sum_\alpha (\lambda_\alpha)^M. \tag{14}$$

Here, Z depends also on the boundary condition of the space lattice (by the $\lambda_\alpha < 1$). In the limit $M,N \to \infty$ it is $Z = A$, where A is the number of attractors (number of $\lambda_\alpha = 1$). All these different boundary conditions and others that are possible generate only surface effects, and as a consequence, for $M,N \to \infty$ these effects should disappear in observable quantities. Indeed, the free energy per site of the space-time lattice is always zero. Further, the later considered n-point correlation functions (n-CF) of the ESM should be independent of the boundary conditions, provided the ESM is defied by a unique Hamiltonian. Thus, the partition function of the ESM can be assumed as

$$Z = \mathrm{Tr}[\exp\{-H(s_t^t)\}], \tag{15}$$

where Tr means summing over all states $s_i^t = 0,1$ of the sites in the space-time lattice. In the derivation of the meaning of the n-CF's, we use the Eq. (14) for Z, because the derivation then is easier, but the result should be independent of this use.

The n-CF are defined in the usual way by

$$\langle s_{i_1}^{t_1}s_{i_2}^{t_2}, \ldots, s_{i_n}^{t_n}\rangle = \frac{1}{Z}\left.\frac{\partial^n Z}{\partial h_{i_1}^{t_1}\partial h_{i_2}^{t_2}, \ldots, \partial h_{i_n}^{t_n}}\right|_{h\to 0}. \tag{16}$$

Substituting Eq. (14) into Eq. (16) a n-CF can be expressed for $t_1 \leq t_2 \leq \ldots \leq t_n$ as

$$\langle s_{i_1}^{t_1} s_{i_2}^{t_2}, \dots, s_{i_n}^{t_n} \rangle = \lim_{M,N\to\infty} \frac{1}{Z(M,N)} \sum_{I_n, \dots, I_1, I} T_{I,I_n}^{M-t_n} (s_{i_n}^{t_n})_{I_n} T_{I_n,I_{n-1}}^{t_n - t_{n-1}} \dots (s_{i_2}^{t_2})_{I_2} T_{I_2,I_1}^{t_2-t_1} (s_{i_1}^{t_1})_{I_1} T_{I_1,I}^{t_1}, \tag{17}$$

where $(s_i^t)_I$ is one if in I the site i at t is active and zero otherwise. Inserting the spectral representation for $T_{I,I_n}^{M-t_n}$ and $T_{I_1,I}^{t_1}$, we get

$$\langle s_{i_1}^{t_1} s_{i_2}^{t_2}, \dots, s_{i_n}^{t_n} \rangle = \frac{1}{A} \sum_{\alpha(\lambda_a=1)} (Q_{i_1,i_2,\dots,i_n}^{t_1,t_2,\dots,t_n})_\alpha, \tag{18}$$

with

$$(Q_{i_1,i_2,\dots,i_n}^{t_1,t_2,\dots,t_n})_\alpha = \lim_{N\to\infty} \sum_{I_n, \dots, I_1} \phi_{\alpha I_n}^L (s_{i_n}^{t_n})_{I_n} T_{I_n,I_{n-1}}^{t_n-t_{n-1}} \dots (s_{i_2}^{t_2})_{I_2} T_{I_2,I_1}^{t_2-t_1} (s_{i_1}^{t_1})_{I_1} \phi_{\alpha I_1}^R. \tag{19}$$

For the case $A=1$ and finite N we have $\phi_{\alpha I_n}^L=1$, thus, each n-CF is the joint probability to observe active sites at (i_1,t_1), (i_2,t_2), $\dots$, (i_n,t_n) if the system is at time t_1 in the unique stationary state. This interpretation must also be true in the limit $N\to\infty$. For the case $A>1$ and $N\to\infty$ the situation is not so clear, but we can argue in the following way: If for finite N the system is not ergodic, there can be more than one $\lambda_a=1$, but then the **T** matrix must disconnect, after an appropriate reordering of the states, into submatrices along the diagonal. Therefore, it seems plausible that an ergodic system with more than one $\lambda_a=1$ for $N\to\infty$ tends to a none-ergodic behavior for very large N, meaning the **T** matrix approximately disconnects and there are $A-1$ eigenvalues extremely near to one. Further, the corresponding $\phi_{\alpha I}^R$ have only large positive values for a specific range of states and the $\phi_{\alpha I}^L$ are large and nearly one for the same range of states. Then for $N\to\infty$ the quantity $(Q_{i_1,i_2,\dots,i_n}^{t_1,t_2,\dots,t_n})_\alpha$ describes the same joint probability as above, but now for the attractor α, and each n-CF is the average of these joint probabilities over all attractors.

For a class of important PCA's an explicit expression for $p(s_i^{t+1}|\mathbf{n}_i^t)$ can be given that leads to an ESM Hamiltonian that has the usual form of Hamiltonians studied in equilibrium statistical mechanics, namely, products of the state variables s_i^t coupled together linearly using some coupling constants. Here we consider only the d-dimensional Domany-Kinzel PCA [4,5]. The synchronous update procedure for this PCA is: If at time t the site i has $n=0,1,\dots,k$ active nearest neighbors, this site is activated at time $t+1$ with probability p_n and inactivated with probability $1-p_n$, regardless of the state of i at t. We define functions $f_n(\mathbf{n}_i^t)=0,1$, where $\mathbf{n}_i^t$ is a state of the set U of nearest neighbors of site i. These functions have the properties that $f_n(\mathbf{n}_i^t)=1$ if exactly n of the nearest neighbors of site i are active and $f_n(\mathbf{n}_i^t)=0$ otherwise. It is easy to see that these f_n can be realized as linear combinations of products of the s_i^t. For example, for $d=1$ it can be chosen $f_0=(1-s_{+1})(1-s_{-1})$, $f_1=s_+(1-s_-)+s_-(1-s_+)$ and $f_2=s_+s_-$, where s_{+1} and s_{-1} are the states of the two neighbors of a site, in general, $f_0=\Pi_{j\in U}(1-s_j^t)$. A valid ESM Hamiltonian is, as can be easily verified, given by

$$H(s_i^t) = -\sum_{i,t} \left(\sum_{n=0}^{k} f_n(\mathbf{n}_i^t)[s_i^{t+1}\ln(p_n) + (1-s_i^{t+1}) \times \ln(1-p_n)] + h_i^t s_i^t \right). \tag{20}$$

This Hamiltonian is the same as the one obtained by Domany and Kinzel [4]. Other valid ESM Hamiltonians can be constructed [15], but this one has a structure that is familiar in equilibrium statistical mechanics. This Hamiltonian is not well defined for $p_n\to 0,1$, but the n-CF's are. A PCA with one absorbing state is obtained for $p_0\to 0$ and $h_i^t=0$. In the space Ω of the p_n with $p_0=0$ and $h_i^t=0$ the only stationary state for points near enough the origin of Ω is the absorbing state with activity zero, and there can be a continuous phase transition at some hypersurface in Ω (critical manifold), so that on the other side of this surface there are two stationary states, the absorbing state and an active state with a mean activity greater than zero. It is known that there is no phase transition for small $p_0>0$ and (or) small $h_i^t\neq 0$ and also not in the limit $p_0\to 0$, unless the limit point is on the critical manifold [4,9].

In equilibrium statistical mechanics for systems with similar Hamiltonians as given in Eq. (20) the scaling properties of correlation functions can be obtained using the renormalization group (RG). Before this is accomplished, the dynamics of the spreading of a perturbation applied to the absorbing state is studied in the framework of the ESM. It is then possible to make a connection between the scaling properties of the n-CF's and the quantities describing the spreading dynamics.

IV. DESCRIPTION OF THE SPREADING DYNAMICS IN THE ESM FRAMEWORK

Let us apply a perturbation to the absorbing state in the form that the sites of a set S are set active at a time t_0. Then an avalanche of activation is spread out for times $t>t_0$. Consider a finite system with $p_0=0$, and denote the state with exactly all sites $i\epsilon S$ active as I_S. Then the evolution of $\mathbf{P}_P(t_0)$ with $P_P(t_0)_{I_S}=1$ and $P_P(t_0)_I=0$ otherwise, as given by Eq. (10), describes the spreading of a perturbation S with m active sites. We are especially interested in the joint probability to find at time $t>t_0$ the sites $i_1, i_2, \ldots, i_n$ active, which is

$$p^t_{i_1,i_2,\ldots,i_n}=\sum_{I,J} (s^t_{i_1}s^t_{i_2}\ldots s^t_{i_n})_I(\mathbf{T}^{t-t_0})_{I,J}P_P(t_0)_J. \tag{21}$$

Our goal is to describe this process in terms of n-CF's of the ESM. To do this, a system with a small p_0 and the stationary state $\boldsymbol{\phi}^R_0(p_0)$ is considered, and finally $p_0\to 0$ is carried out. If it is possible to express the components of $\mathbf{P}_P(t_0)$ as sums over the $\phi^R_0(p_0)_J$, the Eq. (18) can be used to transform the right side of Eq. (21) to n-CF's. One possibility is to choose

$$P_P(t_0)_I=\frac{1}{p_0^m}\prod_{i\epsilon S}(s^{t_0}_i)_I\sum_J T_{I,J}\prod_{j\epsilon S} f_0(\mathbf{n}^{t_0-1}_j)_J\phi^R_0(p_0)_J. \tag{22}$$

To see that this is correct, we note that the factor $\Pi_{j\epsilon S}f_0(\mathbf{n}^{t_0-1}_j)_J\phi^R_0(p_0)_J$ is nonzero only if in the state J for all $j\epsilon S$ all sites from U are inactive. On the other hand, the right side of Eq. (22) is only nonzero for states I with all $i\epsilon S$ active, and to get such an I the $T_{I,J}$ can only generate these active sites using at least m spontaneous activations with probability p_0. Therefore for $p_0\to 0$ the right-hand side of Eq. (22) is one for $I=I_S$ and zero otherwise, where we have used the fact that $\phi^R_0(p_0)_I$ is $1-O(p_0)$ for $I=(0,0,\ldots,0)$ and $O(p_0)$ otherwise. Substituting Eq. (22) into (21) and using Eq. (18), we obtain

$$p^t_{i_1,i_2,\ldots,i_n}=\lim_{p_0\to 0}\frac{1}{p_0^m}\left\langle s^t_{i_1}s^t_{i_2}\ldots s^t_{i_n}\prod_{i\in S}s^{t_0}_i f_0(\mathbf{n}^{t_0-1}_i)\right\rangle_0, \tag{23}$$

where the ESM average $\langle\ldots\rangle_0$ is the part of $\langle\ldots\rangle$ that goes to zero in the limit $p_0\to 0$. This is always the case for finite N, but Eq. (23) is also valid in the limit $N\to\infty$ and for an active phase. Here, the part of $\langle\ldots\rangle$ that goes to a constant for $p_0\to 0$ is not included in $\langle\ldots\rangle_0$. If S consists of one site i_0, the $p^t_{i_1,i_2,\ldots,i_n}$ must be zero if for at least one site i_k with the position vector $\mathbf{r}_{i_k}$ it is $|\mathbf{r}_{i_k}-\mathbf{r}_{i_0}|>t-t_0$. This condition is called the cone condition. For S with more than one site, this condition is more complicated, but it is always ensured by Eq. (23), as can be easily verified.

Summing $p^t_{i_1,i_2,\ldots,i_n}$ over all sites for $n=1$, over all different pairs of sites for $n=2$, over all different triplets of sites for $n=3$, and so on for greater n, we get a quantity called the n-intensity, which is

$$I_n(t)=\frac{1}{n!}\sum_{i_1\neq i_2\neq,\ldots,\neq i_n}p^t_{i_1,i_2,\ldots,i_n}. \tag{24}$$

In spreading experiments [6] $I(t)=I_1(t)$ is measured. Generally, $I_n(t)$ can be measured by cumulating at each time step the possible combinations $\binom{l}{n}$ of the l active sites in the evolution of avalanches. Now it is possible to express the survival probability using only the n-intensities. The quantity

$$Q(t)=\sum_{I,J}\prod_i (1-s^t_i)_I(\mathbf{T}^{t-t_0})_{I,J}P_P(t_0)_J \tag{25}$$

is the probability that at time t there is no activity remaining, meaning the probability that an avalanche has a duration between t_0 and t. Therefore, the survival probability is $P(t)=1-Q(t)$ and it is obtained from Eq. (25) by substituting Eq. (22), using Eqs. (23) and (24) and setting $t_0=0$ for $t>0$ as

$$P(t)=\sum_{n=1}^{\infty}(-1)^{n-1}I_n(t), \tag{26}$$

and $P(0)=1$. Because of the cone condition we must have $I_n(t)=0$ for all $n>n_0(t)\sim t^d$, so that the sum over n is really finite for each finite t. The probability that an avalanche has a duration $t\geqslant 0$ is $p(t)=P(t)-P(t+1)$. The mean duration of an avalanche is defined by $D_A=\Sigma^{\infty}_{t=1}tp(t)$. Using Eq. (26) yields

$$D_A=\sum_{t=1}^{\infty}P(t)=\sum_{n=1}^{\infty}(-1)^{n-1}I_n, \tag{27}$$

where $I_n=\Sigma^{\infty}_{t=1}I_n(t)$ can be interpreted as the total n-intensity.

V. SCALING THEORY OF THE ESM

According to the RG principles, see for example Ref. [20], it is assumed that near the critical manifold for small $p_0>0$ and small h^t_i and provided the h^t_i vary slowly enough over the space-time lattice, a RG transformation exists. Each RG transformation consists of a coarse-graining transformation and a subsequent rescaling to the original lattice spacing and transforms the coupling constants in the ESM Hamiltonian to a new set of coupling constants. We do not construct such a coarse-graining transformation explicitly, but use only general properties of such transformation to derive a scaling theory for the n-CF's. The coarse-graining uses scale factors $a_\perp>1$ in the spatial directions and $a_\parallel>1$ in the time direction. Further, it is assumed that in the limit $p_0\to 0$, a valid RG transformation exists. A sequence of such RG transformations started on the critical manifold has coupling constants that stay on the critical manifold and flow (RG flow) to a unique critical point on this manifold. A sequence of RG transformations that starts near the critical manifold causes a RG flow off the critical manifold. Thus, the critical point is a fixed point of the RG transformation, and there is only one line in Ω $(p_0=0,\ h^t_i=0)$ from which the RG flow starts off the critical point. The distance on this line is measured by $\Delta>0$ if the RG flow goes in the active phase and by $\Delta<0$ otherwise. In the vicinity of the critical point, the RG transformation can be linearized, yielding the linear transformation

$$\tilde{\Delta}=\Lambda_{\Delta}\Delta,\quad \tilde{p}_0=\Lambda_0 p_0,\quad \tilde{h}_i^t=\Lambda_h h_i^t, \tag{28}$$

where Λ_Δ, Λ_0, Λ_h are the eigenvalues of the linearized RG, and it is assumed that there are negligible couplings between the Δ, p_0, and h_i^t directions. The RG flow goes off the critical point in the Δ and h_i^t directions, these directions are relevant in RG terms, therefore $\Lambda_\Delta>1$ and $\Lambda_h>1$. In the p_0 direction, the situation is not so clear, and it turns out later that it is marginal ($\Lambda_0=1$) for the DP class. Because of the group property of the RG, the eigenvalues must have the form

$$\Lambda_\alpha=a_\perp{}^{y_{\perp\alpha}}=a_\parallel^{y_{\parallel\alpha}}, \tag{29}$$

where $\alpha=\Delta,0,h$ and $y_{\perp\alpha},y_{\parallel\alpha}>0$ for relevant directions, and zero for marginal directions. If $\xi(\Delta,p_0)$ (for $h_i^t=0$) is a typical spatial length and $T(\Delta,p_0)$ is a typical time, for example, the correlation length and time of an n-CF, these quantities transform under one RG step as

$$\xi(\tilde{\Delta},\tilde{p}_0)\sim\frac{1}{a_\perp}\xi(\Delta,p_0),\quad T(\tilde{\Delta},\tilde{p}_0)\sim\frac{1}{a_\parallel}T(\Delta,p_0). \tag{30}$$

Using Eqs. (28) and (29), the singular part of ξ and T in the vicinity of the critical point is obtained from Eq. (30) as

$$\xi(\Delta,p_0)\sim\frac{1}{|\Delta|^{\nu_\perp}}f_\perp(p_0|\Delta|^{-y_{\perp 0}\nu_\perp}),$$

$$T(\Delta,p_0)\sim\frac{1}{|\Delta|^{\nu_\parallel}}f_\parallel(p_0|\Delta|^{-y_{\parallel 0}\nu_\parallel}), \tag{31}$$

with $\nu_\perp=1/y_{\perp\Delta}$ and $\nu_\parallel=1/y_{\parallel\Delta}$. The scaling of a general n-CF

$$G_n(\mathbf{r}_{i1},t_i,\Delta,p_0,h_i^t)=\langle s_{j_1}^{u_1}s_{j_2}^{u_2},\ldots,s_{j_n}^{u_n}\rangle, \tag{32}$$

where $u_1\leqslant u_2\leqslant\ldots\leqslant u_n$, $\mathbf{r}_{i1}=\mathbf{r}_{j_{i+1}}-\mathbf{r}_{j_1}$, $t_i=u_{i+1}-u_1$ with $i=1,2,\ldots,n-1$ for $n>1$, and where for $n=1$ no $\mathbf{r}_i$, t_i dependence occurs, can be derived from Eq. (16) and the fact that Z is invariant under an RG transformation. The result is

$$G_n\left(\frac{r_{kl}}{a_\perp},\frac{t_i}{a_\parallel},\tilde{\Delta},\tilde{p}_0,\tilde{h}_i^t\right)\sim\frac{a_\perp{}^{nd}a_\parallel{}^n}{\Lambda_h{}^n}G_n(r_{kl},t_i,\Delta,p_0,h_i^t), \tag{33}$$

where $r_{kl}=|\mathbf{r}_{j_k}-\mathbf{r}_{j_l}|\gg a_\perp$ for all $k,l=1,2,\ldots,n$ with $k>l$ and $t_i\gg a_\parallel$ for all $i=1,2,\ldots,n-1$. In a similar way, as Eq. (31) is obtained from Eq. (30), one finds that the scaling properties of G_n for $\Delta\neq0$ and $h_i^t=0$ are

$$G_n(r_{kl},t_i,\Delta,p_0)\sim|\Delta|^{\alpha_n}f_n^{\pm}(r_{kl}|\Delta|^{\nu_\perp},t_i|\Delta|^{\nu_\parallel},p_0|\Delta|^{-y_{\perp 0}\nu_\perp}), \tag{34}$$

where $\alpha_n=n[\nu_\parallel+\nu_\perp(d-y_{\perp h})]$ and f_n^+ is for $\Delta>0$, f_n^- for $\Delta<0$. Because there is no phase transition for $\Delta\neq0$ and $p_0\to0$, it is possible to expand Eq. (34) in a power series of $p_0\Delta^{-y_{\perp 0}\nu_\perp}$, where the term of degree m reads

$$G_{n,m}(r_{kl},t_i,\Delta,p_0)\sim p_0^m|\Delta|^{\alpha_n-my_{\perp 0}\nu_\perp}f_{n,m}^{\pm}(r_{kl}|\Delta|^{\nu_\perp},t_i|\Delta|^{\nu_\parallel}). \tag{35}$$

The quantity $G_{1,0}$ equals for $p_0\to0$ the constant mean activity $\langle s_i^t\rangle$, which is greater than zero for $\Delta>0$, therefore, from Eq. (35) we get $\langle s_i^t\rangle\sim\Delta^\beta$ with $\beta=\nu_\parallel+\nu_\perp(d-y_{\perp h})$. There is an upper critical spatial dimension d_c, so that for $d\geqslant d_c$ the mean-field values of the critical exponents are valid. For the DP universality class it is $d_c=4$, and $\nu_\perp=1/2$, $\nu_\parallel=1$, $\beta=1$ are the mean-field values [1], therefore for $d=4$ we have $y_{\perp h}=4$. For $d>4$ a scaling relation that includes d (hyperscaling relation) cannot be valid. This is the consequence of the fact that for $d\geqslant4$ the above scaling theory with the relevant or marginal scaling variables Δ, h, p_0, and all other scaling variables are irrelevant cannot be true, at least one of the irrelevant variables must be a dangerous one [20], meaning that scaling functions such as $f_n^{\pm}$ of Eq. (34) are not analytical in such a variable and this variable cannot be ignored. We do not further investigate the consequences and restrict ourselves to $d\leqslant4$. An important variant of Eq. (35) is

$$G_{n,m}(r_{kl},t_i,\Delta,p_0)\sim\frac{p_0^m}{t_j^{(n\beta-my_{\perp 0}\nu_\perp)/\nu_\parallel}}f_{n,m}(r_{kl}/t_j^{z/2},t_i^{1/\nu_\parallel}\Delta), \tag{36}$$

with $z=2\nu_\perp/\nu_\parallel$ and t_j being any of the t_i. This expression gives also the scaling form for $\Delta=0$. If some of the $s_{j_k}^{u_k}$ in Eq. (32) are replaced by clusters $\Pi_{(i,u)\epsilon V_k}s_i^u$, where V_k is a set of sites around the site (j_k,u_k) with an extension of the order $a_\perp$ in spatial direction and $a_\parallel$ in time direction, and the distances between these clusters are $r_{kl}\gg a_\perp$ and $t_j\gg a_\parallel$, the resulting n-CF must obey the same scaling as given by Eqs. (34), (35), and (36).

Now we are ready to get the scaling properties of the $p_{i_1,i_2,\ldots,i_n}^t$ given by Eq. (23) for the case that the set S is a set of sites around (i_0,t_0) with an spatial extension of at most the order $a_\perp$. The right-hand side of Eq. (23) can be expressed as a sum over certain n-CF's and each of these terms has the form $\langle s_{i_1}^t s_{i_2}^t,\ldots,s_{i_n}^t\Pi_{(i,u)\in S_0}s_i^u\rangle$, where S_0 is a set that has the properties of the above V_k sets, therefore the scaling of each term can be obtained from Eq. (36) if $t-t_0\gg a_\parallel$ and $r_{kl}=|\mathbf{r}_{i_k}-\mathbf{r}_{i_l}|\gg a_\perp$ for $k,l=0,1,\ldots,n$, $k>l$. By replacing each $s_{i_k}^t$ by the average over the s_j^t belonging to sites in a small spatial volume around i_k, the corresponding sum over the $p_{i_1,i_2,\ldots,i_n}^t$ is the averaged n-density $\rho_n(r_{kl},t)$ of activations, which obey the scaling

$$\rho_n(r_{kl},t)\sim\frac{1}{t^{[(n+1)\beta-my_{\perp 0}\nu_\perp]/\nu_\parallel}}g_{n,m}\left(\frac{r_{kl}}{t^{z/2}},t^{1/\nu_\parallel}\Delta\right), \tag{37}$$

with $n=1,2,\ldots$, and $g_{n,m}$ is a scaling function proportional to $f_{n+1,m}$. It is not possible to realize the cone condition exactly with Eq. (37), therefore, Eq. (37) is only valid within the cone somewhat apart from the cone boundaries. Consider the case $m=2$ and the two perturbed sites are nearest neighbors, then for the Domany-Kinzel PCA it is easy to see that the spreading of activation evolves independently in the same way on two different intertwined i,t sublattices [4]. This means that the n-density for the case $m=2$ can differ from the n-density for $m=1$ only by a constant factor and cannot have different critical exponents as indicated by Eq. (37), thus, it follows $y_{\perp 0}=0$. Therefore, only three independent critical exponents exist, for example $\nu_\parallel$, $\nu_\perp$, and $y_{\perp h}$, and it must be possible to express all other exponents by scaling relations using only a set of three independent exponents. This result is in accordance with other derivation [1,6,7]. Because we are interested mainly in scaling properties, it is sufficient to consider only the case $m=1$.

Integrating Eq. (37) over all spatial coordinates gives the n-intensity at t as

$$I_n(t)\sim t^{\eta+(n-1)\eta_1}\phi_n(t^{1/\nu_\parallel}\Delta), \tag{38}$$

where $\eta=dz/2-2\beta/\nu_\parallel$ and $\eta_1=\eta+\beta/\nu_\parallel$. For $\Delta=0$ the n intensity should increase with t, therefore $\eta>0$ and $\eta_1>\eta>0$, and for $d\geqslant 4$ it is $\eta=0$ and $\eta_1=1$. In the case $\Delta<0$ the n-intensity decays exponentially for $t\to\infty$ with a correlation time $T_n\sim|\Delta|^{-\nu_\parallel}$. The leading part of the total n intensity is obtained for $\Delta<0$ by integrating Eq. (38) over t, which yields

$$I_n\sim\frac{c_n}{|\Delta|^{\gamma+(n-1)\nu_\parallel\eta_1}},\quad \gamma=\nu_\parallel(1+\eta). \tag{39}$$

VI. SURVIVAL PROBABILITY OF ACTIVITY AVALANCHES

In the asymptotic limit $t\to\infty$, the survival probability $P(t)$ is obtained by inserting the scaling form Eq. (38) of $I_n(t)$ into Eq. (26). For fixed t, the infinite sum in Eq. (26) is really finite, and the scaling form of $I_n(t)$ is only valid for $t\gg(n/2)^{1/d}$, thus, in the asymptotic limit all terms must be considered, and we get

$$P(t)\sim t^{\eta}\sum_{n=0}^{\infty}(-1)^n(t^{\eta_1})^n\phi_{n+1}(t^{1/\nu_\parallel}\Delta). \tag{40}$$

Consider first $\Delta=0$. In this case, the series in Eq. (40) is a power series $f(x)$ of $x=t^{\eta_1}$, and for $t\to\infty$ we must have $P(t)\sim 1/t^{\delta}$ with $0<\delta\leqslant 1$. Therefore, there is a $x_0>0$ such that for $x>x_0$ $f(x)=x^{-k}\ \tilde{f}(x)$, with $k>0$ and $\lim_{x\to\infty}\tilde{f}(x)>0$. Assuming that $\tilde{f}(x)$ can be expanded in a power series leads to the conclusion that this must be also true for $x^k f(x)$. Because it is not possible to expand x^k in a unique power series for all $x>x_0$ if k not an integer, so we conclude that k must be an integer. Thus, $\delta+\eta-k\eta_1=0$, and for $d=4$ we have $\delta=k$, so that only $k=1$ is possible. Assuming that k is independent of d gives

$$P(t)\sim\frac{1}{t^{\delta}},\quad \delta=\eta_1-\eta=\beta/\nu_\parallel. \tag{41}$$

Substituting $\beta/\nu_\parallel=\delta$ into the scaling relation for η leads to $4\delta+2\eta=dz$. This hyperscaling relation was first derived by Grassberger and de la Torre [7] using the assumption that for $\Delta>0$ and large t, a sphere with nearly constant ρ_1 is expanding with a constant velocity v, see also Ref. [1]. A further derivation was recently given using a continuous field theory of the PCA [10].

Using the last result we get for $\Delta\neq 0$

$$P(t)\sim\frac{1}{t^{\delta}}\sum_{n=1}^{\infty}(-1)^{n-1}(t^{\eta_1})^n\phi_n(t^{1/\nu_\parallel}\Delta). \tag{42}$$

The series in Eq. (42) has a complicated structure, and it is not obvious how we could get the usually assumed scaling form Eq. (1). But we have already shown that really no such scaling exists. For $\Delta>0$, $P(t)$ must approach to the constant value $P(\infty)=\rho_1/\langle s_i^t\rangle$. From Eq. (37) it follows $\rho_1=\Delta^{2\beta}f(r\Delta^{\nu_\perp})$ for $t\to\infty$ and assuming that for large t there is a d-dimensional sphere with constant ρ_1 expanding with constant velocity, see Ref. [7], we have $\rho_1\sim\Delta^{2\beta}$, thus $P(\infty)\sim\Delta^{\beta}=\Delta^{\delta\nu_\parallel}$. The behavior of $P(t)$ for $\Delta<0$ and large t depends on the behavior of the infinite sum in Eq. (42), but this is a complicated task. To get more insight, we study the mean duration of avalanches $D_A=\int_0^\infty tp(t)dt$. Inserting Eq. (39) into Eq. (27) gives

$$D_A\sim\frac{1}{|\Delta|^{\gamma}}\sum_{n=0}^{\infty}(-1)^n\frac{c_{n+1}}{(|\Delta|^{\eta_1\nu_\parallel})^n}. \tag{43}$$

The infinite series in Eq. (43) is a power series in $1/|\Delta|^{\eta_1\nu_\parallel}$. There are three possibilities concerning the functional behavior of $D_A(\Delta)$. The first possibility is $D_A\sim 1/|\Delta|^{\kappa}$. Assuming that the series in Eq. (43) is convergent for $\Delta\neq 0$ and using arguments similar as above to get Eq. (41), the asymptotic behavior of Eq. (43) is obtained as

$$D_A\sim\frac{1}{|\Delta|^{\kappa}},\quad \kappa=\gamma-l\,\eta_1\nu_\parallel, \tag{44}$$

where l is an integer. Using the scaling relations for γ, η_1, and δ we get $\kappa=[1+\eta-l(\delta+\eta)]\nu_\parallel$. For $d=4$ we have $\kappa=1-l$ and to get a $\kappa\geqslant 0$ only $l\leqslant 1$ is possible. Choosing $l\leqslant 0$ gives $\kappa\geqslant\nu_\parallel$, where the equal sign is only valid for $d=4$ and $l=0$, and for $l=1$ we have $\kappa=\nu_\parallel(1-\delta)<\nu_\parallel$. Setting $l=1$, we have $\kappa=(1-\delta)\nu_\parallel$, this was derived by Miguel *et al.* [6] using the scaling assumption. The second possibility is $D_A\sim f(\Delta)$, where $f(\Delta)$ is a complicated function with $\lim_{\Delta=0}f(\Delta)=\infty$. The last possibility is that the series in Eq. (43) does not converge in a narrow range $|\Delta|<\Delta_c$ around $\Delta=0$, meaning $D_A=\infty$ in this range. Strictly speaking, the series in Eq. (43) cannot go to infinity in this case, but the series $D_A=\Sigma_{t=1}^{\infty}P(t)$ [see Eq. (27)] then is also divergent and gives $D_A=\infty$. If the first or second possibility is realized, a serious drawback of the universality of scaling arises, because Eq. (31) should be valid also for D_A. This can be only realized if $D_A\sim|\Delta|^{-\kappa}$, but it is not possible to choose the integer l in Eq. (44) so that $\kappa=\nu_\parallel$ for all d. The last possibility is the most appealing one. The series in Eq. (43) diverges if the c_n approach for $n\to\infty$ not fast enough to zero, that means precisely $\lim_{n\to\infty}c_n^{1/n}>0$. If we assume this, then there will exist such a Δ_c so that in the range $|\Delta| < \Delta_c$, the mean duration is $D_A = \infty$. The consequence of this assumption is that for $|\Delta| < \Delta_c$, we must have $p(t,\Delta) \sim h(\Delta)/t^{1+\delta}$, where $\lim_{\Delta \to 0} h(\Delta) > 0$ and $h(\Delta)$ is analytical around $\Delta = 0$. We admit that this possibility looks very strange, but it gives for $|\Delta| < \Delta_c$ a unified scaling view: Equation (30) can be satisfied for all quantities for which it should be valid, and if $h(\Delta)$ varies only slightly around $\Delta = 0$, this can be interpreted as a correction to scaling. Note that if really $\lim_{n\to\infty} c_n^{1/n} > 0$, the total n-intensities I_n must grow exponentially for $n \to \infty$ and $|\Delta| < \Delta_c$, see Eq. (39).

For the avalanche sizes, the situation is different because the mean size is $S_A = I_1$, so that $S_A \sim |\Delta|^{-\gamma}$. The consequence is that $p(s,\Delta) \sim f(s,\Delta)/s^{\tau}$, with $f(s,\Delta)$ analytical in Δ, $\lim_{s=\infty} f(s,\Delta) = 0$ for $\Delta \neq 0$ and such that the correct asymptotic behavior of S_A results. Indeed, assuming that $P(t) \sim 1/t^{\delta}$ for $|\Delta| < \Delta_c$, it is possible to construct such a $f(s,\Delta)$: Following the path of derivation given in [6] to get the critical exponent $\tau = (1+\eta+2\delta)/(1+\eta+\delta)$, the average size of an avalanche with duration t is $S_A(t) \sim \int^t dt' I_1(t')/P(t')$. Inserting for I_1 Eq. (38) and $P(t) \sim 1/t^{\delta}$ we get $S_A(t) \sim t^{1+\eta+\delta} \hat{\phi}(t^{1/\nu_\parallel}\Delta)$. Assuming that the conditional probability $p(s|t)$ for an avalanche having size s, given it dies at time t, is $F(u)$ with $u = s/S_A(t)$ and $F(u)$ is bell shaped, with its maximum at $u = 1$, we get $p(s) \sim \int^{\infty} dt F(u)/[S_A(t) t^{1+\delta}]$. This $p(s)$ is not a scaling function, but calculating S_A gives the correct result $S_A \sim |\Delta|^{-\gamma}$. Further it is easy to see that this $p(s)$ has a cutoff s_c, which is the maximal s value giving $u = 1$, and we get $s_c \sim |\Delta|^{-1/\sigma}$, where $1/\sigma = \nu_\parallel(1+\eta+\delta)$ in agreement with Ref. [6] . On the other hand, if there is not a critical region with $P(t) \sim 1/t^{\delta}$ it is quite unclear how the correct asymptotic behavior of S_A could be obtained.

Outside the critical region defined by $|\Delta| < \Delta_c$ it seems that the conventionally assumed scaling behavior of the discussed quantities is approximately valid. Inspecting the results of spreading experiments presented in the literature [19,21], the above hypothesis about the existence of such a critical region looks not very appealing. Looking at simulation results for $P(t,\Delta)$ drawn in a double-logarithmic P vs t plot seems to show for $\Delta < 0$ and large t approximately an exponential behavior. According to our hypothesis, these curves should make a crossing over from a slope larger δ to a constant slope δ for large t, supposing Δ is sufficiently close to zero. If this behavior really exists, it must reside outside the presently used maximal avalanche duration in such spreading experiments.

There is another indication, obtained from a simulation experiment, which supports the above view of the scaling problem. If there is a lower bound $J_n \leq I_n$ of I_n with $\lim_{n\to\infty} J_n^{1/n} > 0$ for some fixed Δ near $\Delta = 0$, we conclude from Eqs. (39) and (43) the existence of a critical region with $D_A = \infty$. Such a J_n could be the part of I_n in which exactly n and not more sites are active. For a finite system with N sites we have $J_n(N) \leq J_n \leq I_n$. Therefore, in spreading experiments with a fixed $\Delta < 0$ and system size N, the counting of the number of events with exactly n active sites gives $J_n(N)$. For fixed N and larger $n \leq N$ it is difficult to get $J_n(N)$ values with low enough statistical fluctuations, but enlarging for growing n the system size, for example $N = nM$ with M fixed, it is possible to get for relatively large n trustable $J_n(nM)$ values. Trustable values for $I_n(N) = \Sigma_{l=n}^{N} \binom{l}{n} J_l(N)$ are difficult to obtain, so we hope that the lower bound $J_n(nM)$ can show the desired behavior from a plot of $J_n(nM)^{1/n}$ versus $1/n$. This plot must clearly indicate that it is possible to extrapolate $J_n(nM)^{1/n}$ for $1/n \to 0$ to a value $C > 0$ and the possibility $C = 0$ can be excluded with high enough certainty. We have done spreading experiments for the one-dimensional Domany-Kinzel PCA with $p_1 = p_2 = 0.7054$, which corresponds to $\Delta = -0.000\,085\,3$. Figure 2 shows plots of $J_n(nM)^{1/n}$ as a function of $1/n$ for different M values, from below to above it is $M = 8$ with $n_{\max} = 1250$ and $M = 16$ and 32 with $n_{\max} = 2000$. The statistical error of these data is lesser than the extension of the dots indicating the data points. The extrapolating curves are obtained from a fit of the data points with a polynomial of degree three. The curve for $M = 32$ should be very near the limit curve for $N \to \infty$. All data points are near one and far from zero, and n goes up to very high values, indicating that it seems very unlikely that this curve can ever go to zero for $1/n \to 0$, but that is not proof.

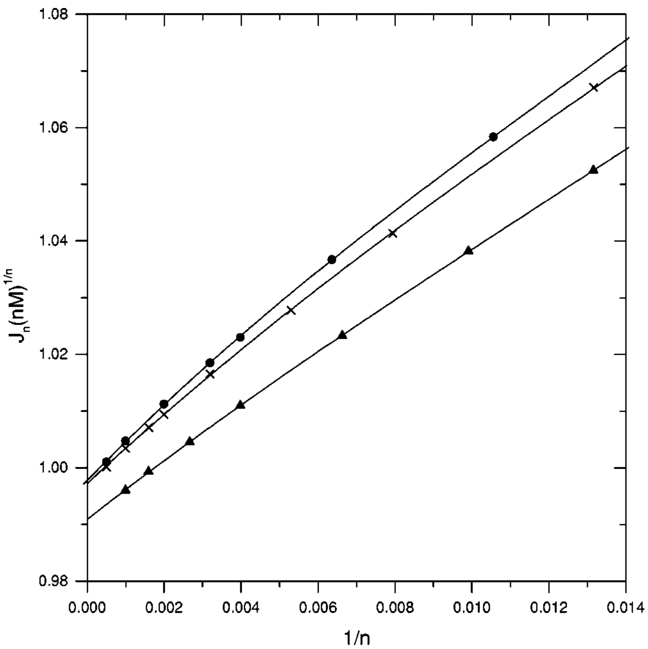

FIG. 2. Behavior of a lower bound of the n-intensity for high n values (up to $n = 2000$) for a one-dimensional Domany-Kinzel PCA with the size $N = nM$ and $M = 8$,16, and 32 (from below to above) in the inactive phase near the critical point for $p_1 = p_2 = 0.7054$.

VII. CONCLUSIONS

We have shown that the probability density $p(t,\Delta)$, the survival probability $P(t,\Delta)$, and the size density $p(s,\Delta)$ of avalanches in the absorbing state of PCA's from the DP universality class cannot obey the usually assumed asymptotic scaling behavior. The origin for this anomalous behavior is the fact that $p(t,\Delta)$ is expandable as a power series in Δ and that the condition $\int_0^\infty p(t,\Delta)dt + P(\infty,\Delta) = 1$ must be valid for all Δ. Therefore, the question arises what is the correct asymptotic behavior of these quantities near the phase transition. To investigate this point, we have used an equilibrium statistical mechanic model (ESM) for a Domany-Kinzel PCA in d dimensions. Using the scaling properties of certain correlation functions of the ESM it is possible to express the mean duration D_A of avalanches as an infinite series of certain inverse power-law terms. From this result for D_A we can get some hints about the asymptotic behavior of D_A for $\Delta \to 0$. The first possibility is that this behavior is very complicated and not accessible at present. Second assuming $D_A \sim |\Delta|^{-\kappa}$, we have shown that κ cannot be equal to the critical exponent $\nu_\parallel$. These two possibilities are not very tempting, because they are not in accordance with scaling theory. But as shown, there is another possible solution, namely, that the infinite series for D_A does not converge near $\Delta = 0$. In this case we must have $D_A = \infty$ and $p(t,\Delta) \sim f(\Delta)/t^{1+\delta}$ for $|\Delta| < \Delta_c$, where Δ_c is a new very small critical value. For this strange looking hypothesis we have obtained some evidence: First, scaling theory is true at least for $|\Delta| < \Delta_c$. Second, we are able to construct a $f(s,\Delta)$, which is not a scaling function, so that $p(s,\Delta) \sim f(s,\Delta)/s^\tau$, and the mean size of avalanches is $S_A \sim |\Delta|^{-\gamma}$, as it must be. Finally we have undertaken a special simulation experiment on a one-dimensional Domany-Kinzel PCA that supports strongly the Δ_c hypothesis. Outside the critical region the usually assumed scaling forms of the avalanche quantities could be valid approximately. It seems to be very difficult to exactly prove the Δ_c hypothesis. Spreading experiments do not seem to be able to reach large enough avalanche durations for which the postulated crossing over of the survival probability to an inverse power-law behavior could be observed. That means that only further theoretical investigation can solve the scaling problem with certainty. In the case that it will turn out that the Δ_c hypothesis is true, this may imply important consequences for natural systems that operate near a phase transition, because then catastrophic events that exhibit large duration should be much more likely as conventionally assumed. Especially stochastic systems that show the so-called ‘‘self-organized to critically’’ phenomenon [22] should be presumably effected.

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